Some differential complexes within and beyond parabolic geometry

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Dedicated to Professors Reiko Miyaoka and Keizo Yamaguchi on their sixtieth birthdays

Abstract.

For smooth manifolds equipped with various geometric structures, we construct complexes that replace the de Rham complex in providing an alternative fine resolution of the sheaf of locally constant functions. In case that the geometric structure is that of a parabolic geometry, our complexes coincide with the Bernstein-Gelfand-Gelfand complex associated with the trivial representation. However, at least in the cases we discuss, our constructions are relatively simple and avoid most of the machinery of parabolic geometry. Moreover, our method extends to contact and symplectic geometries (beyond the parabolic realm).

§1. Introduction

In [8], Čap, Slovák, and Souček construct sequences of invariant differential operators on *parabolic geometries* of any type G/P, one for each finite-dimensional representation $\mathbb{V}$ of G. (Here, G is a semisimple Lie group and $P \subset G$ a parabolic subgroup.) These sequences are known as *Bernstein-Gelfand-Gelfand* (BGG) sequences since, for the homogeneous model G/P of such a geometry, these sequences are complexes, which are dual to a parallel construction due to these authors [2] on the level of Verma modules. In [5] Calderbank and Diemer simplify the construction of BGG sequences in [8]. In addition they provide [5, p. 87], for regular parabolic geometries, alternative BGG sequences, which only coincide with the ones in [8] if the geometry is *torsion-free*. The latter sequences

not only appear to be more natural, they also have the advantage that if $\mathbb{V}$ is taken to be the trivial representation, then they form complexes, providing fine resolutions of the locally constant sheaf $\mathbb{R}$ (as one sees by suitably modifying [5, Proposition 5.5(iv)]). For the sequences of [8] this is only true if the geometry is torsion-free and, in this case, the two sequences are anyway the same. In combination with the construction of canonical Cartan connections given in [6], this shows that one can find alternatives to the de Rham resolution for any parabolic geometry defined in terms of a regular infinitesimal flag structure [7, §3.1.6]. A hallmark of these resolutions is that the ranks of the bundles involved are diminished as compared to the de Rham complex. The price one pays is that the operators may be higher than first order. The construction of these resolutions in [5, 8], entails firstly constructing the Cartan connection as described in [6] and this is not at all straightforward.

In this article we present some examples constructed by a more elementary route. As we show, our method extends to certain non-parabolic geometries, namely arbitrary contact and symplectic geometries. We shall use the spectral sequence of a filtered complex [10] without comment and merely as a replacement for tedious diagram chasing.

§2.   The Rumin complex

For our first example we shall construct the Rumin complex [15]. It is defined on an arbitrary contact manifold but, for simplicity, we shall present the 5-dimensional case, which is typical. So let M be a 5-dimensional smooth manifold with $H \subset TM$ a contact distribution. Equivalently, the contact structure may be defined by $L \equiv H^{\perp}$, a line sub-bundle of the bundle of 1-forms Λ^1. If we define a rank 4 vector bundle Λ_H^1 as the quotient Λ^1/L, then there are induced short exact sequences

$$0 \to \Lambda_H^{p-1} \otimes L \to \Lambda^p \to \Lambda_H^p \to 0, \quad \text{for } 1 \leq p \leq 5$$

and the spectral sequence of the de Rham complex filtered in this way reads, at the E_0-level,

$$
\begin{array}{cccccc}
q & \Lambda^0 & \Lambda_H^1 & \Lambda_H^2 & \Lambda_H^3 & \Lambda_H^4 & 0 \\
 & & & \mathcal{L} \uparrow & \uparrow & \uparrow & \\
 & & 0 & L & \Lambda_H^1 \otimes L & \Lambda_H^2 \otimes L & \Lambda_H^3 \otimes L & \Lambda_H^4 \otimes L \\
 & p & & & & &
\end{array}
$$

where $\mathcal{L}$ is the composition $L \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda_H^2$. The Leibniz rule shows that $\mathcal{L}$ is linear over the functions and is, therefore, a homomorphism of vector bundles. It is called the *Levi form*. By definition

of contact manifold, the range of $\mathcal{L}$ is non-degenerate as a skew form on H, defined up to scale. Equivalently, we can choose local co-framings $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ with ω_1 a section of L such that

$$d\omega_1 \equiv \omega_2 \wedge \omega_3 + \omega_4 \wedge \omega_5 \bmod \omega_1.$$

Notice that

$$d(\omega_1 \wedge \omega_2) \equiv \omega_2 \wedge \omega_4 \wedge \omega_5 \bmod \omega_1 \qquad d(\omega_1 \wedge \omega_3) \equiv \omega_3 \wedge \omega_4 \wedge \omega_5 \bmod \omega_1$$
$$d(\omega_1 \wedge \omega_4) \equiv \omega_2 \wedge \omega_3 \wedge \omega_4 \bmod \omega_1 \qquad d(\omega_1 \wedge \omega_5) \equiv \omega_2 \wedge \omega_3 \wedge \omega_5 \bmod \omega_1$$

whence the E_0-differential $\Lambda^1_H \otimes L \to \Lambda^3_H$ is an isomorphism of vector bundles. Similar reasoning shows that $\Lambda^2_H \otimes L \to \Lambda^4_H$ is surjective. Hence, at the E_1-level we obtain

$$
\begin{array}{c}
q \\
\quad \Lambda^0 \xrightarrow{d_\perp} \Lambda^1_H \xrightarrow{d_\perp} \Lambda^2_{H\perp} \quad 0 \qquad 0 \qquad 0 \\
\quad\quad\quad 0 \qquad 0 \qquad 0 \quad \Lambda^2_{H\perp} \otimes L \xrightarrow{d_\perp} \Lambda^3_H \otimes L \xrightarrow{d_\perp} \Lambda^4_H \otimes L \\
p
\end{array}
$$

and deduce that there is a complex

$$(1)\ 0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d_\perp} \Lambda^1_H \xrightarrow{d_\perp} \Lambda^2_{H\perp} \xrightarrow{d^{(2)}_\perp} \Lambda^2_{H\perp} \otimes L \xrightarrow{d_\perp} \Lambda^3_H \otimes L \xrightarrow{d_\perp} \Lambda^4_H \otimes L \to 0$$

where $\Lambda^p_{H\perp}$ denotes the sub-bundle of Λ^p_H, trace-free with respect to the Levi-form. The operator $d^{(2)}_\perp$ is second order and, because the spectral sequence converges to the local cohomology of the de Rham complex, it follows that this complex is exact on the level of sheaves. Already, the Rumin complex goes beyond parabolic geometry. Notice that, although a convenient co-framing was chosen to perform some calculation, the construction itself and the resulting complex are independent of any such choice. This is a repeated theme in this article.

§3. The Engel complex

In this section we shall be concerned with a smooth 4-manifold M equipped with a generic distribution $H \subset TM$ of rank 2. Genericity entails that $[H, H]$ has rank 3 and that $[H, [H, H]] = TM$. Dually, if we let $K \equiv H^\perp$ and $L \equiv [H, H]^\perp$ then the 1-forms are filtered $L \subset K \subset \Lambda^1$ by the line-bundle L and rank 2 bundle K. In fact, there is a canonically defined finer filtration constructed as follows. One easily checks that the Levi form $K \to \Lambda^2_H$, defined as the composition $K \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H$,

is a surjective homomorphism of vector bundles with L as kernel. It follows that the other Levi form, defined as the composition

$$L \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2(\Lambda^1/L)$$

has range in the kernel of $\Lambda^2(\Lambda^1/L) \to \Lambda^2_H$. However, the short exact sequence

$$0 \to K/L \to \Lambda^1/L \to \Lambda^1_H \to 0$$

identifies this kernel as $\Lambda^1_H \otimes K/L$. In other words, we have a canonically defined inclusion $L \otimes (K/L)^* \hookrightarrow \Lambda^1_H$ the range of which defines a line sub-bundle ξ of Λ^1_H. The result is that we can write

$$\Lambda^1 = \Lambda^1_H/\xi + \xi + K/L + L,$$

meaning that Λ^1 is filtered with composition factors being line bundles as indicated (ordered so that Λ^1_H/ξ is a canonical quotient and L is a canonical sub-bundle). All in all, if we write λ for Λ^1_H/ξ and untangle the identifications found above, then we conclude that

$$(2) \qquad\qquad \Lambda^1 = \lambda + \xi + \lambda\xi + \lambda\xi^2.$$

Equivalently, we can work locally with $(\omega^1, \omega^2, \omega^3, \omega^4)$, an *adapted* co-framing such that

$$(3) \qquad d\omega^1 \equiv \omega^2 \wedge \omega^3 \bmod \omega^1 \quad \text{and} \quad d\omega^2 \equiv \omega^3 \wedge \omega^4 \bmod \omega^1, \omega^2,$$

noting that the freedom in such a co-framing comprises exactly the triangular endomorphisms of the filtration (2), where

$$L = \lambda\xi^2 = \mathrm{span}\{\omega^1\}, \quad K = \lambda\xi + \lambda\xi^2 = \mathrm{span}\{\omega^1, \omega^2\},$$
$$\xi + K = \mathrm{span}\{\omega^1, \omega^2\, \omega^3\}.$$

So far, this is the structure of an *Engel manifold*. As with the Rumin complex, it is clear that the first order operator $d_H : \Lambda^0 \to \Lambda^1_H$ defined as the composition $\Lambda^0 \xrightarrow{d} \Lambda^1 \to \Lambda^1_H$ has the locally constant functions as its kernel. We now seek differential conditions on a section of Λ^1_H in order that it be in the range of the operator d_H. Starting with any 1-form ω,

(4)
- define f by $d\omega \equiv f\,\omega^3 \wedge \omega^4 \bmod \omega^1, \omega^2$
- define p by $d(\omega - f\omega^2) \equiv p\,\omega^2 \wedge \omega^4 + g\,\omega^2 \wedge \omega^3 \bmod \omega^1.$

The structure equations (3) show that p is well-defined and one easily checks that the equivalence class

$$[p\,\omega^2 \wedge \omega^4] \in \frac{\mathrm{span}\{\omega^2 \wedge \omega^4, \omega^2 \wedge \omega^3, \omega^1 \wedge \omega^4, \omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2\}}{\mathrm{span}\{\omega^2 \wedge \omega^3, \omega^1 \wedge \omega^4, \omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2\}} \cong \lambda^2\xi$$

depends only on the equivalence class $[\omega] \in \Lambda_H^1$ and is independent of choice of co-framing. We have a well-defined second order differential operator

$$\Lambda_H^1 \ni [\omega] \xmapsto{\mathcal{P}} [p\,\omega^2 \wedge \omega^4] \in \lambda^2 \xi,$$

giving what we shall call the *primary* obstruction to $[\omega]$ being in the range of d_H. In a chosen co-frame, one can easily proceed to find a *secondary* obstruction s as follows. Define f, p, g by (4) and then

• define s by $d(\omega - f\omega^2 - g\omega^1) = p\,\omega^2 \wedge \omega^4 + r\,\omega^1 \wedge \omega^4 + s\,\omega^1 \wedge \omega^3 + t\,\omega^1 \wedge \omega^2$.

If p vanishes, then

$$0 = d^2(\omega - f\omega^2 - g\omega^1) = r\,\omega^2 \wedge \omega^3 \wedge \omega^4 + \cdots$$

so r vanishes. If, in addition s vanishes, then

$$0 = d^2(\omega - f\omega^2 - g\omega^1) = d(t\,\omega^1 \wedge \omega^2) = -t\,\omega^1 \wedge \omega^3 \wedge \omega^4$$

so t vanishes. Hence, if both p and s vanish, then $d(\omega - f\omega^2 - g\omega^1) = 0$. By the Poincaré Lemma, it follows that $[\omega]$ is locally in the range of d_H, as required. If the primary obstruction vanishes, then the equivalence class

$$[s\,\omega^1 \wedge \omega^3] \in \frac{\text{span}\{\omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2\}}{\text{span}\{\omega^1 \wedge \omega^2\}} \cong \lambda\xi^3$$

is independent of choice of co-framing. Otherwise, the change

(5) $$\omega_4 \mapsto \omega_4 + h\omega_3$$

induces severe complications with s changing by r and its derivatives. If one wants to avoid these complications, it suffices to prohibit (5) to arrive at an invariantly defined differential operator

$$(\mathcal{P}, \mathcal{S}) : \Lambda_H^1 \to \lambda^2 \xi \oplus \lambda\xi^3,$$

whose kernel is locally the range of d_H. More precisely, we may eliminate (5) by choosing a complement to the line sub-bundle $\xi \hookrightarrow \Lambda_H^1$. In other words, we choose a splitting $\Lambda_H^1 = \lambda \oplus \xi$. An adapted co-framing yields such a splitting and, conversely, a fixed choice of splitting restricts the choice of adapted co-framings precisely by preventing the addition of any multiple of ω^3 to ω^4. The forms on an Engel manifold endowed with this extra structure are filtered as follows.

$$\begin{aligned}
\Lambda^1 &= (\lambda \oplus \xi) + \lambda\xi + \lambda\xi^2 \\
\Lambda^2 &= \lambda\xi + (\lambda^2\xi \oplus \lambda\xi^2) + (\lambda^2\xi^2 \oplus \lambda\xi^3) + \lambda^2\xi^3 \\
\Lambda^3 &= \lambda^2\xi^2 + \lambda^2\xi^3 + (\lambda^3\xi^3 \oplus \lambda^2\xi^4)
\end{aligned}$$

and the spectral sequence of the de Rham complex filtered in this way reads, at the E_0-level,

$$
\begin{array}{cccccccc}
\Lambda^0 & \lambda \oplus \xi & \lambda\xi & 0 & 0 & 0 & 0 \\
& & \uparrow & & & & \\
& 0 & \lambda\xi & \lambda^2\xi \oplus \lambda\xi^2 & \lambda^2\xi^2 & 0 & 0 \\
& & & \uparrow & \uparrow & & \\
& 0 & 0 & \lambda\xi^2 & \lambda^2\xi^2 \oplus \lambda\xi^3 & \lambda^2\xi^3 & 0 \\
& & & & \uparrow & & \\
& 0 & 0 & 0 & 0 & \lambda^2\xi^3 & \lambda^3\xi^3 \oplus \lambda^2\xi^4 & \Lambda^4 \,.
\end{array}
$$

The E_0-differentials are easily computed in our adapted co-frame. For example

$$f\,\omega^1 \wedge \omega^2 \xrightarrow{\ d\ } f\,\omega^1 \wedge \omega^3 \wedge \omega^4 \bmod \omega^1 \wedge \omega^2$$

$$h\,\omega^1 \wedge \omega^4 + g\,\omega^1 \wedge \omega^3 + f\,\omega^1 \wedge \omega^2 \xrightarrow{\ d\ } h\,\omega^2 \wedge \omega^3 \wedge \omega^4 \bmod \omega^1 \wedge \omega^2,\, \omega^1 \wedge \omega^3 \wedge \omega^4$$

deals with the two rightmost differentials. Consequently, at the E_1-level we obtain

$$
\begin{array}{cccccccc}
\Lambda^0 & \lambda \oplus \xi & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & \lambda^2\xi & 0 & 0 & 0 \\
& 0 & 0 & 0 & \lambda\xi^3 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & \lambda^3\xi^3 \oplus \lambda^2\xi^4 & \Lambda^4 \,.
\end{array}
$$

The bundles $\lambda \oplus \xi$ and $\lambda^3\xi^3 \oplus \lambda^2\xi^4$ may be identified with Λ^1_H and $\Lambda^1_H \otimes \Lambda^2 K$, respectively. The line bundles $\lambda^2\xi$ and $\lambda\xi^3$ combine to give a rank 2 vector bundle $\lambda^2\xi + \lambda\xi^3$ but, in fact, this bundle canonically splits as can readily be seen in our adapted co-frame:

$$\lambda^2\xi + \lambda\xi^3 = \frac{\mathrm{span}\{\omega^2 \wedge \omega^4, \omega^1 \wedge \omega^2, \omega^1 \wedge \omega^4\}}{\mathrm{span}\{\omega^1 \wedge \omega^2, \omega^1 \wedge \omega^4\}} \oplus \frac{\mathrm{span}\{\omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2\}}{\mathrm{span}\{\omega^1 \wedge \omega^2\}},$$

independent of choice of co-frame. We conclude that there is a complex of differential operators (cf. [14])

$$(6) \qquad \Lambda^0 \xrightarrow{\ d_H\ } \Lambda^1_H \to \lambda^2\xi \oplus \lambda\xi^3 \to \Lambda^1_H \otimes \Lambda^2 K \to \Lambda^4 \to 0$$

resolving the locally constant sheaf $\mathbb{R}$. Following through the spectral sequence more explicitly as a diagram chase shows that $\Lambda^1_H \to \lambda^2\xi \oplus \lambda\xi^3$ is given by our previous recipe. We shall see later in §7, that (6) is a BGG complex for an appropriate parabolic geometry.

§4. The Rumin complex revisited

Since a contact manifold with no extra structure is not a parabolic geometry, the Rumin complex lies outside the realm of parabolic geometry. Nevertheless, there is a parabolic geometry in which the Rumin complex finds its genesis. Let us denote by $\mathrm{Sp}(2n, \mathbb{R})$ the simple Lie group of linear automorphisms of $\mathbb{R}^{2n}$ preserving a fixed non-degenerate symplectic form. Viewing the $(2n+1)$-sphere S^{2n+1} as

$$\{x \in \mathbb{R}^{2n+2} \text{ s.t. } x \neq 0\}/\{x \sim \lambda x \text{ for } \lambda > 0\}$$

(i.e. the space of rays emanating from the origin in $\mathbb{R}^{2n+2}$), the group $G = \mathrm{Sp}(2n+2, \mathbb{R})$ acts smoothly and transitively on S^{2n+1}. The stabiliser subgroup P of this action is parabolic. Parabolic geometries modelled on this particular homogeneous space $S^{2n+1} = G/P$ are known as *contact projective* [7, §4.2.6]. In any case, when viewed in this way, the sphere S^{2n+1} inherits a G-invariant contact structure from the symplectic form on $\mathbb{R}^{2n+2}$. As in §2, let us now consider the case $n = 2$. Adopting the notation from [1], this homogeneous space is written as ✕━━●⟺● and the Bernstein-Gelfand-Gelfand complex corresponding to the trivial representation of $\mathrm{Sp}(6, \mathbb{R})$ is

$$0 \to \overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!\Longleftrightarrow\!\!\overset{0}{\bullet} \to \overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!\Longleftrightarrow\!\!\overset{0}{\bullet} \to \overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!\Longleftrightarrow\!\!\overset{0}{\bullet} \to \overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!\Longleftrightarrow\!\!\overset{1}{\bullet}$$

$$\to \overset{-5}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!\Longleftrightarrow\!\!\overset{1}{\bullet} \to \overset{-6}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!\Longleftrightarrow\!\!\overset{0}{\bullet} \to \overset{-6}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!\Longleftrightarrow\!\!\overset{0}{\bullet} \to 0.$$

This coincides with the Rumin complex (1). The reason for the notation is fully explained in [1]. Here, suffice it to say that

$$\overset{q}{\bullet}\!\!\Longleftrightarrow\!\!\overset{r}{\bullet}(\Lambda_H^1) \otimes L^s = \overset{-2s-2q-3r}{\times}\!\!-\!\!\overset{q}{\bullet}\!\!\Longleftrightarrow\!\!\overset{r}{\bullet}$$

where $\overset{q}{\bullet}\!\!\Longleftrightarrow\!\!\overset{r}{\bullet}(\Lambda_H^1)$ denotes the bundle induced by the irreducible representation $\overset{q}{\bullet}\!\!\Longleftrightarrow\!\!\overset{r}{\bullet}$ of $\mathrm{Sp}(4, \mathbb{R})$ (meaning that its highest weight is $[q, r]$ with respect to the standard Bourbaki-ordered basis of fundamental weights).

In summary, there is a homogeneous contact geometry G/P, with G simple and P parabolic, for which the BGG complex coincides with the Rumin complex.

§5. Pfaffian systems of rank three in five variables

Let M be a 5-manifold equipped with $H \subset TM$, a generic distribution of rank 2. Equivalently, let $I \subset \Lambda^1$ be a Pfaffian system of rank 3 that is generic in Cartan's sense, i.e. the first derived system I' has rank

2 and the second derived system I'' is zero. We have a filtration of the tangent bundle

$$H \subset [H, H] \subset TM \quad \text{by vector bundles of ranks} \quad 2, 3, 5$$

and a dual filtration of the cotangent bundle, which we shall write as

$$\Lambda^1 = \Lambda^1_H + L + I', \tag{7}$$

where L is the line-bundle I/I'. There are locally defined co-framings $(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ so that the following congruences hold

$$\begin{gathered} d\omega^1 \equiv \omega^3 \wedge \omega^4 \bmod \omega^1, \omega^2 \qquad d\omega^2 \equiv \omega^3 \wedge \omega^5 \bmod \omega^1, \omega^2 \\ d\omega^3 \equiv \omega^4 \wedge \omega^5 \bmod \omega^1, \omega^2, \omega^3 \end{gathered} \tag{8}$$

with $I' = \operatorname{span}\{\omega^1, \omega^2\}$ and $I = \operatorname{span}\{\omega^1, \omega^2, \omega^3\}$. We shall refer to such co-framings as *adapted*. The Levi form for I, defined as the composition $I \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H$, has I' as its kernel (by definition of I' or by viewing this form in an adapted co-frame). Hence, the line-bundle L may be canonically identified with Λ^2_H. Similarly, the Levi form for I', namely $I' \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H + \Lambda^1_H \otimes L$, canonically identifies I' with $\Lambda^1_H \otimes L$. Therefore, we may rewrite (7) as

$$\Lambda^1 = \Lambda^1_H + \Lambda^2_H + \Lambda^1_H \otimes \Lambda^2_H. \tag{9}$$

To proceed, it is useful to have a more compact notation for the bundles induced by Λ^1_H. Following a common convention for the irreducible representations of $\operatorname{GL}(2, \mathbb{C})$, let us write

$$(a, b) \in \mathbb{Z}^2 \text{ with } a \le b \text{ for the bundle } \odot^{b-a} \Lambda^1_H \otimes (\Lambda^2_H)^a, \tag{10}$$

where $\odot$ means symmetric tensor product. Then (9) becomes
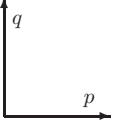
$$\Lambda^1 = (0, 1) + (1, 1) + (1, 2)$$

and the induced filtration on 2-forms is

$$\Lambda^2 = (1, 1) + (1, 2) + \begin{matrix} (1, 3) \\ \oplus \\ (2, 2) \end{matrix} + (2, 3) + (3, 3).$$
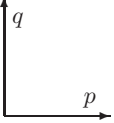
Without further ado, we may now consider the spectral sequence of the
de Rham complex filtered in this way. At the E_0-level we obtain

$$
\begin{array}{cccccccc}
\Lambda^0 & (0,1) & (1,1) & 0 & 0 & 0 & & \\
 & & \uparrow & & & & & \\
 & 0 & (1,1) & (1,2) & (2,2) & 0 & 0 & \\
 & & & \uparrow & \uparrow & & & \\
 & & 0 & (1,2) & (1,3)\oplus(2,2) & (2,3) & 0 & 0 \\
 & & & & & \uparrow & & \\
 & & & 0 & 0 & (2,3) & (2,4)\oplus(3,3) & (3,4) & 0 \\
 & & & & & & \uparrow & \uparrow \\
 & & & & 0 & 0 & (3,3) & (3,4) & (4,4) & 0 \\
 & & & & & & & & \uparrow \\
 & & & & & 0 & 0 & 0 & (4,4) & (4,5) & \Lambda^5.
\end{array}
$$

(with q vertical axis and p horizontal axis)

The E_0-level differentials are easily computed from the structure equa-
tions (8), the E_1-level is

$$
\begin{array}{ccccccccc}
\Lambda^0 & \to & (0,1) & 0 & 0 & 0 & 0 & & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & (1,3) & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & (2,4) & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & (4,5) & \to & (5,5),
\end{array}
$$

(with q vertical axis and p horizontal axis)

and we have shown that there is a differential complex

$$
\Lambda^0 \xrightarrow{\nabla} \Lambda^1_H \xrightarrow{\nabla^3} (1,3) \xrightarrow{\nabla^2} (2,4) \xrightarrow{\nabla^3} (4,5) \xrightarrow{\nabla} (5,5)
$$

resolving the constant sheaf $\mathbb{R}$, where ∇^k simply indicates a differential
operator of order k. If necessary, the structure equations (8) can be used
to compute the operators precisely. To compare with the usual BGG
complex, we follow Cartan [9] in realising the flat model for this geometry
as a homogeneous space G/P where G is the exceptional non-compact
Lie group G_2 and P is a parabolic subgroup. Specifically, following the
notation of [1], the homogeneous space is ⋊⟹ . The Levi factor of the
parabolic subgroup is $\mathrm{GL}(2,\mathbb{R})$ but it is useful to identify its Lie algebra
with $\mathfrak{g}_0$ where we have graded the Lie algebra of G_2

$$
\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}_{\mathfrak{p}}
$$

in accordance with the parabolic subalgebra $\mathfrak{p}$ (see [7]). With these conventions, the cotangent bundle is

$$\Lambda^1 = \overset{-2\ \ 1}{\rlap{\times}{\Longleftarrow}} + \overset{-1\ \ 0}{\rlap{\times}{\Longleftarrow}} + \overset{-3\ \ 1}{\rlap{\times}{\Longleftarrow}} \tag{11}$$

and the BGG complex is

$$\overset{0\ \ 0}{\rlap{\times}{\Longleftarrow}} \overset{\nabla}{\longrightarrow} \overset{-2\ \ 1}{\rlap{\times}{\Longleftarrow}} \overset{\nabla^3}{\longrightarrow} \overset{-5\ \ 2}{\rlap{\times}{\Longleftarrow}} \overset{\nabla^2}{\longrightarrow} \overset{-6\ \ 2}{\rlap{\times}{\Longleftarrow}} \overset{\nabla^3}{\longrightarrow} \overset{-6\ \ 1}{\rlap{\times}{\Longleftarrow}} \overset{\nabla}{\longrightarrow} \overset{-5\ \ 0}{\rlap{\times}{\Longleftarrow}}. \tag{12}$$

More generally, in Dynkin diagram notation the bundle (a,b) is written as $\overset{a-2b\ \ b-a}{\rlap{\times}{\Longleftarrow}}$.

In fact, there are several other complexes that can be created from the de Rham complex by choosing to carry out only some of the diagram chasing involved in creating the BGG complex. We now explain two of these complexes and their motivation. Keeping the Dynkin diagram notation, the filtration of the 2-forms induced from (11) is

$$\Lambda^2 = \overset{-1\ \ 0}{\rlap{\times}{\Longleftarrow}} + \overset{-3\ \ 1}{\rlap{\times}{\Longleftarrow}} + \begin{matrix} \overset{-5\ \ 2}{\rlap{\times}{\Longleftarrow}} \\ \oplus \\ \overset{-2\ \ 0}{\rlap{\times}{\Longleftarrow}} \end{matrix} + \overset{-4\ \ 1}{\rlap{\times}{\Longleftarrow}} + \overset{-3\ \ 0}{\rlap{\times}{\Longleftarrow}},$$

which suggests that one might cancel $\overset{-1\ \ 0}{\rlap{\times}{\Longleftarrow}} + \overset{-3\ \ 1}{\rlap{\times}{\Longleftarrow}}$ from the exterior derivative $\Lambda^1 \overset{d}{\to} \Lambda^2$. But, as a sub-bundle of Λ^1, this is precisely the original Pfaffian system I. So, we are trying to cancel from $\Lambda^1 \to \Lambda^2$, the homomorphism defined as the composition

$$I \hookrightarrow \Lambda^1 \overset{d}{\to} \Lambda^2 \to \frac{\Lambda^2}{\Lambda^1 \wedge I'}.$$

We can accomplish this explicitly by means of an adapted co-frame. Specifically, we define a differential operator

$$\mathcal{E} : \Lambda^1_H = \Lambda^1/I \to \Lambda^1 \wedge I' \subset \Lambda^2 \tag{13}$$

by the following steps. Starting with any 1-form ω,

- define f by $d\omega \equiv f\,\omega^4 \wedge \omega^5 \bmod \omega^1, \omega^2, \omega^3$,
- define g, h by $d(\omega - f\omega^3) \equiv g\,\omega^3 \wedge \omega^4 + h\,\omega^3 \wedge \omega^5 \bmod \omega^1, \omega^2$.

This is possible according to the structure equations (8), which also imply that

$$d(\omega - f\omega^3 - g\omega^1 - h\omega^2) \equiv 0 \bmod \omega^1, \omega^2,$$

in other words that

$$\mathcal{E}\omega \equiv d(\omega - f\omega^3 - g\omega^1 - h\omega^2) \in \Lambda^1 \wedge I' \subset \Lambda^2.$$

One checks easily that this definition of $\mathcal{E}\omega$ is independent of choice of adapted co-framing. Moreover, if ω is actually a section of I, say $\omega = F\omega^3 + G\omega^1 + H\omega^2$, then $f = F$, $g = G$, and $h = H$, whence $\mathcal{E}\omega = 0$. In other words, the differential operator $\mathcal{E}$ descends to Λ^1_H, as claimed in (13).

Theorem 1. *The sequence*

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{\mathcal{E}} \Lambda^1 \wedge I' \xrightarrow{d} \Lambda^3 \xrightarrow{d} \Lambda^4 \xrightarrow{d} \Lambda^5 \to 0$$

is a locally exact complex.

Proof. This is just a matter of unravelling definitions, bearing in mind that the de Rham complex is itself locally exact. Suppose, for example, that ω is a 1-form representing a section of Λ^1_H that is annihilated by $\mathcal{E}$. Locally, we need to find a smooth function ϕ such that $\omega - d\phi$ is a section of I. By construction of $\mathcal{E}$ we know $d(\omega - f\omega^3 - g\omega^1 - h\omega^2) = 0$ for some smooth functions f, g, h. Thus, by exactness of the de Rham complex, locally we can write $\omega - f\omega^3 - g\omega^1 - h\omega^2 = d\phi$ and then $\omega - d\phi = f\omega^3 + g\omega^1 + h\omega^2$ is a section of I, as required. The remaining verifications are similarly straightforward. Q.E.D.

The operator $\mathcal{E} : \Lambda^1_H \to \Lambda^1 \wedge I'$ has a geometric meaning: a section ϕ of Λ^1_H can be regarded as a Lagrangian for an integral curve of I. From this point of view $\mathcal{E}\phi$ are the Euler-Lagrange equations associated to this Lagrangian. From its construction, one can easily verify that $\mathcal{E}$ is third order. More specifically, by construction, its symbol

$$\bigodot^3 \Lambda^1 \otimes \Lambda^1_H \to \Lambda^1 \wedge I' = \begin{array}{c} \overset{-5\ 2}{\rightrightarrows} \\ \oplus \\ \overset{-2\ 0}{\rightrightarrows} \end{array} + \overset{-4\ 1}{\rightrightarrows} + \overset{-3\ 0}{\rightrightarrows}$$

composes with the projection to $\overset{-5\ 2}{\rightrightarrows}$ as the homomorphism

$$\bigodot^3 \Lambda^1 \otimes \Lambda^1_H = \bigodot^3 \big(\overset{-2\ 1}{\rightrightarrows} + \overset{-1\ 0}{\rightrightarrows} + \overset{-3\ 1}{\rightrightarrows} \big) \otimes \overset{-2\ 1}{\rightrightarrows}$$
$$\to \bigodot^3 \big(\overset{-2\ 1}{\rightrightarrows} \big) \otimes \overset{-2\ 1}{\rightrightarrows} \to \bigodot^2 \big(\overset{-2\ 1}{\rightrightarrows} \big) \otimes \Lambda^2 \big(\overset{-2\ 1}{\rightrightarrows} \big)$$
$$= \overset{-4\ 2}{\rightrightarrows} \otimes \overset{-1\ 0}{\rightrightarrows} = \overset{-5\ 2}{\rightrightarrows}.$$

Furthermore, not only does the symbol have no component in $\overset{-2\ 0}{\rightrightarrows}$ but, in fact, the range of the operator $\mathcal{E}$ is entirely contained in the sub-bundle where this component vanishes. This is easily seen in an adapted co-frame: since

$$d(\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^4) = 2\,\omega^3 \wedge \omega^4 \wedge \omega^5 \bmod \omega^1, \omega^2$$

and since $\omega^1 \wedge \omega^5 - \omega^2 \wedge \omega^4$ spans $\overset{-2\quad 0}{\Longleftarrow}$, any exact 2-form in $\Lambda^1 \wedge I'$ has vanishing component in $\overset{-2\quad 0}{\Longleftarrow}$. If we denote by B^2 the rank 6 sub-bundle of $\Lambda^1 \wedge I'$ defined as the kernel of the natural projection $\Lambda^1 \wedge I' \to \overset{-2\quad 0}{\Longleftarrow}$, and by B^3 the rank 9 sub-bundle of Λ^3 generated by ω^1, ω^2, then we have cancelled $\overset{-2\quad 0}{\Longleftarrow}$ from the complex of Theorem 1 and demonstrated the following improvement.

Theorem 2. *The sequence*

$$(14) \qquad 0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{\mathcal{E}} B^2 \xrightarrow{d} B^3 \xrightarrow{d} \Lambda^4 \xrightarrow{d} \Lambda^5 \to 0$$

is a locally exact complex.

The ranks of the bundles and the orders of the differential operators in (14) are

$$1 \xrightarrow{\nabla} 2 \xrightarrow{\nabla^3} 6 \xrightarrow{\nabla} 9 \xrightarrow{\nabla} 5 \xrightarrow{\nabla} 1$$

but if we consider $d : B^2 \to B^3$ in more detail

$$B^2 = \overset{-5\ \ 2}{\Longleftarrow} + \overset{-4\ \ 1}{\Longleftarrow} + \overset{-3\ \ 0}{\Longleftarrow} \to \overset{-4\ \ 1}{\Longleftarrow} + \begin{matrix} \overset{-6\ \ 2}{\Longleftarrow} \\ \oplus \\ \overset{-3\ \ 0}{\Longleftarrow} \end{matrix} + \overset{-5\ \ 1}{\Longleftarrow} + \overset{-4\ \ 0}{\Longleftarrow} = B^3,$$

then it suggests that we should be able to eliminate $\overset{-4\ \ 1}{\Longleftarrow} + \overset{-3\ \ 0}{\Longleftarrow}$ from both bundles. This is, indeed, the case as can be seen in an adapted co-frame: writing a general section of B^2 as

$$\omega = \mu \wedge \omega^1 + \nu \wedge \omega^2 \text{ s.t. } d\omega \equiv 0 \bmod \omega^1, \omega^2$$

we may define a differential operator $\mathcal{F} : B^2 \to B^3$ by the following familiar steps.

- Define f, g by $d\omega \equiv\ f\,\omega^1 \wedge \omega^4 \wedge \omega^5 + g\,\omega^2 \wedge \omega^4 \wedge \omega^5$
  $\mod \omega^1 \wedge \omega^2, \omega^1 \wedge \omega^3, \omega^2 \wedge \omega^3,$
- Define h by $d(\omega - f\,\omega^1 \wedge \omega^3 - g\,\omega^2 \wedge \omega^3) \equiv$
  $h\,(\omega^2 \wedge \omega^3 \wedge \omega^4 - \omega^1 \wedge \omega^3 \wedge \omega^5)$
  $+ p\,\omega^1 \wedge \omega^3 \wedge \omega^4 + q\,\omega^2 \wedge \omega^3 \wedge \omega^5$
  $+ r\,(\omega^2 \wedge \omega^3 \wedge \omega^4 + \omega^1 \wedge \omega^3 \wedge \omega^5)$
  $\mod \omega^1 \wedge \omega^2.$

This is possible according to the structure equations (8), which also imply that

$$\mathcal{F}\omega \equiv d(\omega - f\,\omega^1 \wedge \omega^3 - g\,\omega^2 \wedge \omega^3 - h\,\omega^1 \wedge \omega^2)$$

lies in the sub-bundle

$$C^3 \equiv \overset{-6\ \ 2}{\Longleftarrow} + \overset{-5\ \ 1}{\Longleftarrow} + \overset{-4\ \ 0}{\Longleftarrow}$$

of $B^3 \subset \Lambda^3$ and that it descends to the quotient

$$B^2 = \overset{-5\;\;2}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-4\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-3\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \twoheadrightarrow \overset{-5\;\;2}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \equiv C^2$$

of B^2. It is easily verified that this definition of $\mathcal{F}$ is independent of choice of co-framing and that, if we denote by $\bar{\mathcal{E}}$ the composition

$$\Lambda^1_H \xrightarrow{\;\mathcal{E}\;} B^2 \to C^2,$$

then the expected theorem follows:

Theorem 3. *The sequence*

$$(15) \qquad 0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{\;\bar{\mathcal{E}}\;} C^2 \xrightarrow{\;\mathcal{F}\;} C^3 \xrightarrow{\;d\;} \Lambda^4 \xrightarrow{\;d\;} \Lambda^5 \to 0$$

is a locally exact complex.

The ranks of the bundles and the orders of the differential operators in (15) are

$$1 \xrightarrow{\nabla} 2 \xrightarrow{\nabla^3} 3 \xrightarrow{\nabla^3} 6 \xrightarrow{\nabla} 5 \xrightarrow{\nabla} 1.$$

Writing (15) as

$$\overset{0\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \to \overset{-2\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \to \overset{-5\;\;2}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \to \overset{-6\;\;2}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-5\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-4\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \to$$
$$\overset{-5\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-4\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-6\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} \to \overset{-5\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet},$$

suggests one final cancellation, specifically of $\overset{-5\;\;1}{\rlap{\times}{\Longleftarrow}\!\!\bullet} + \overset{-4\;\;0}{\rlap{\times}{\Longleftarrow}\!\!\bullet}$ from C^3 and Λ^4. The reader can readily verify that this gives the BGG complex (12). It is interesting to note that the ranks of the bundles and orders of differential operators in the BGG complex are

$$1 \xrightarrow{\nabla} 2 \xrightarrow{\nabla^3} 3 \xrightarrow{\nabla^2} 3 \xrightarrow{\nabla^3} 2 \xrightarrow{\nabla} 1.$$

In particular, the order of the differential operator in the middle has gone down from 3 to 2. Since our filtering on the de Rham complex is, by construction, compatible with the tautological Hodge isomorphisms $\Lambda^p = \Lambda^5 \otimes (\Lambda^{5-p})^*$, and since we have run the spectral sequence to its end, it follows that the BGG complex is formally self-adjoint.

§6. Pfaffian systems of rank three in six variables

Let M be a 6-manifold equipped with $H \subset TM$, a generic distribution of rank 3. Equivalently, let $I \subset \Lambda^1$ be a Pfaffian system of rank 3 that is generic in Cartan's sense, i.e. the first derived system I' is zero.

Locally there are co-framings $(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6)$ so that $\omega^1, \omega^2, \omega^3$ span I and the following congruences hold.

(16)
$$d\omega^1 \equiv \omega^5 \wedge \omega^6 \bmod \omega^1, \omega^2, \omega^3 \quad d\omega^2 \equiv \omega^6 \wedge \omega^4 \bmod \omega^1, \omega^2, \omega^3$$
$$d\omega^3 \equiv \omega^4 \wedge \omega^5 \bmod \omega^1, \omega^2, \omega^3.$$

In the terminology of [4], these co-framings are 1-*adapted*. As usual, let us write Λ^1_H for Λ^1/I. Then the Levi form $\mathcal{L} : I \to \Lambda^2_H$ defined as the composition $I \hookrightarrow \Lambda^1 \xrightarrow{d} \Lambda^2 \twoheadrightarrow \Lambda^2_H$ is an isomorphism and we can canonically identify I with Λ^2_H as vector bundles. Indeed, this isomorphism is apparent in our 1-adapted co-framing (16). We may mimic (10) to write, up to isomorphism, the general Schur-irreducible bundle induced by Λ^1_H as $(a, b, c) \in \mathbb{Z}^3$ with $a \leq b \leq c$ for the bundle

$$(\odot^{c-b}\Lambda^1_H \otimes \odot^{b-a}(\Lambda^1_H)^*)_\circ \otimes (\Lambda^3_H)^b,$$

where $\circ$ as a subscript means to take the trace-free part. These observations mean that we may write the filtration

$$\Lambda^1 = \Lambda^1_H + I \quad \text{as} \quad \Lambda^1 = (0, 0, 1) + (0, 1, 1)$$

and decompose the induced filtrations on the higher forms as

$$\Lambda^2 = (0, 1, 1) + \begin{matrix} (0, 1, 2) \\ \oplus \\ (1, 1, 1) \end{matrix} + (1, 1, 2)$$

(17)
$$\Lambda^3 = (1, 1, 1) + \begin{matrix} (0, 2, 2) \\ \oplus \\ (1, 1, 2) \end{matrix} + \begin{matrix} (1, 1, 3) \\ \oplus \\ (1, 2, 2) \end{matrix} + (2, 2, 2)$$

$$\Lambda^4 = (1, 2, 2) + \begin{matrix} (1, 2, 3) \\ \oplus \\ (2, 2, 2) \end{matrix} + (2, 2, 3) \quad \Lambda^5 = (2, 2, 3) + (2, 3, 3)$$

$$\Lambda^6 = (3, 3, 3).$$

From the structure equations (16) for a 1-adapted co-framing it is easily verified that all expected cancellations at the E_0-level of the associated spectral sequence actually take place and we have proved the following result.

Theorem 4. *There is a canonically defined locally exact differential complex*

$$0 \to \mathbb{R} \to (0, 0, 0) \to (0, 0, 1) \to (0, 1, 2) \to \big[(0, 2, 2) + (1, 1, 3)\big] \to$$
$$(1, 2, 3) \to (2, 3, 3) \to (3, 3, 3) \to 0$$

on any smooth 6-manifold equipped with a generic 3-distribution.

For the moment, the bundle $\big[(0,2,2) + (1,1,3)\big]$ is a canonically defined sub-quotient of Λ^3 but, in fact, one can improve matters as the following theorem shows (the analogous step was not necessary in §5).

Theorem 5. *A splitting of the short exact sequence*

$$0 \to (0,1,1) \to \Lambda^1 \to (0,0,1) \to 0$$

gives rise to a homomorphism of vector bundles defined as the composition

$$(18) \qquad\qquad (1,1,1) \to \Lambda^2 \xrightarrow{\;d\;} \Lambda^3 \to (0,2,2)$$

and there is a preferred class of splittings characterised by requiring that this induced homomorphism vanish. This preference canonically splits the bundle $\big[(0,2,2) + (1,1,3)\big]$.

Proof. Certainly, a splitting of the 1-forms splits all the other forms and so, from (17), one may consider the composition (18) obtained by splitting the 2-forms and 3-forms. To see that it is a homomorphism, rather than the differential operator it might appear to be, notice that if Ω is a 2-form in $(1,1,1)$, then

$$(19) \qquad\qquad d(f\Omega) = f\,d\Omega + df \wedge \Omega$$

and it is clear that $df \wedge \Omega$ has components only in

$$\Lambda^1 \otimes (1,1,1) = \big((0,0,1) \oplus (0,1,1)\big) \otimes (1,1,1) = (1,1,2) \oplus (1,2,2)$$

inside

$$\Lambda^3 = (1,1,1) \oplus \begin{matrix} (0,2,2) \\ \oplus \\ (1,1,2) \end{matrix} \oplus \begin{matrix} (1,1,3) \\ \oplus \\ (1,2,2) \end{matrix} \oplus (2,2,2).$$

In particular, if we project to $(0,2,2)$ as in (18), then $df \wedge \Omega$ does not contribute and, from (19), the result is linear over the functions. Now suppose we change the splitting of the 1-forms. The freedom in doing so lies in

$$(20) \qquad \mathrm{Hom}\big((0,0,1),(0,1,1)\big) = (-1,1,1) \oplus (0,0,1).$$

This same freedom shows up in splitting the first part of Λ^3:

$$\Lambda^3 = (1,1,1) + \begin{matrix} (0,2,2) \\ \oplus \\ (1,1,2) \end{matrix} + \cdots$$

and

$$\mathrm{Hom}\left((1,1,1),\ \begin{matrix}(0,2,2)\\ \oplus\\ (1,1,2)\end{matrix}\right)=\begin{matrix}(-1,1,1)\\ \oplus\\ (0,0,1)\end{matrix}.$$

Bearing in mind that the composition $(1,1,1)\to\Lambda^2\to\Lambda^3\to(1,1,1)$ is an isomorphism, independent of choice of splitting (it is responsible for one of the cancellations occurring at the E_0-level of the spectral sequence), we conclude that we can spend the $(-1,1,1)$-freedom in splitting precisely in setting the homomorphism (18) to zero. Now let us consider how this impacts on the sub-quotient $\big[(0,2,2)+(1,1,3)\big]$ of Λ^3. The freedom in splitting this sub-quotient lies in

(21) $\mathrm{Hom}\big((0,2,2),(1,1,3)\big)=(-1,-1,3)\oplus(-1,0,2)\oplus(-1,1,1)$

and one sees that the only way that (20) can enter is through $(-1,1,1)$. Having eliminated this freedom by a preferred choice of splittings for Λ^1, it is thereby eliminated from (21) and we have obtained our canonical splitting.   Q.E.D.

The preferred splittings of Λ^1 afforded by Theorem 5 can be conveniently expressed in terms of our 1-adapted co-framings satisfying (16). If such a co-framing is used to split Λ^1, then the resulting sub-bundle $(1,1,1)$ of Λ^2 is spanned by $\Omega\equiv\omega^1\wedge\omega^4+\omega^2\wedge\omega^5+\omega^3\wedge\omega^6$ and, following through its proof, the preferred splittings of Theorem 5 are characterised by requiring that

$$d\Omega\equiv 3\,\omega^4\wedge\omega^5\wedge\omega^6+\omega\wedge\Omega\ \mathrm{mod}\ \omega^1\wedge\omega^2,\omega^2\wedge\omega^3,\omega^3\wedge\omega^1,\quad\text{for some 1-form }\omega.$$

In the terminology of [4], co-framings satisfying this extra congruence are called 2-*adapted*.

The Lie algebra $\mathfrak{so}(4,3)$ admits a grading of the form

(22)
$$\underbrace{\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0}_{\mathfrak{p}}\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2=$$

and one can see from this grading that the corresponding 6-dimensional homogeneous space G/P is equipped with a canonical 3-dimensional distribution. The corresponding infinitesimal flag structure [7, §3.1.6] is exactly the geometry of such 3-distributions and the irreducible bundles are related in our two notations by

$$(a,b,c)=\ \overset{b-a\quad c-b\quad -2c}{\bullet\!\!-\!\!\!-\!\!\!\Longrightarrow\!\!\!\times}.$$

Recall in the proof of Theorem 5 that we reduced the freedom in splitting Λ^1 to $(0,0,1)$ in (20). In the Dynkin diagram notation this remaining freedom lies in $\overset{0\ \ \ 1\ \ -2}{\bullet\!-\!\bullet\!\Rrightarrow\!\times}$, which is exactly the action of $\mathfrak{g}_1$ on

$$\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \overset{0\ \ \ 1\ \ \ 0}{\bullet\!-\!\bullet\!\Rrightarrow\!\times} \oplus \overset{1\ \ \ 0\ \ \ 0}{\bullet\!-\!\bullet\!\Rrightarrow\!\times},$$

as can be seen in (22). The geometric import of this observation is that Theorem 5 reduces the structure group of the tangent bundle from general H-preserving and Levi-form-preserving automorphisms to the subgroup of $\mathrm{Aut}(\mathfrak{g}/\mathfrak{p})$ defined by the Adjoint action of P, namely the group $P/\exp(\mathfrak{g}_2)$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. Dually, the 2-adapted co-framings are preserved by exactly this group.

Finally, we can take the complex of Theorem 4, use the splitting of $\big[(0,2,2)+(1,1,3)\big]$ afforded by Theorem 5, and write the result in Dynkin diagram notation to obtain the following.

Theorem 6. *On any smooth 6-manifold equipped with a generic 3-distribution, there is a canonically defined locally exact differential complex*

This is the BGG complex in standard notation.

§7. The Engel complex revisited

Although the complex $\Lambda^0 \xrightarrow{d_H} \Lambda^1_H \xrightarrow{\mathcal{P}} \lambda\xi^2$ constructed in §3 used nothing beyond an Engel structure, for the full-blown resolution (6) it was necessary to choose some extra structure, namely a complement to $\xi \subset \Lambda^1_H$ (equivalently, a complement to $(\xi + K)^\perp \subset H$, the *Engel line field* [12]). As pointed out to us by Boris Doubrov, there is a unique homogeneous space of the form G/P, for G semisimple and P parabolic, that carries a G-invariant Engel structure. Specifically, if $G = \mathrm{Sp}(4,\mathbb{R})$ and P is its Borel subgroup, then

$$\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}_{\mathfrak{p}},$$

which, in Dynkin diagram notation, reads

$$\overset{2\quad 0}{\bullet\!=\!\!=\!\!\Rightarrow\!\!\times} = \overset{2\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times} \oplus \overset{0\quad 1}{\times\!\!=\!\!\Rightarrow\!\!\times} \oplus \begin{matrix}\overset{2\quad -1}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus \\ \underset{-2\quad 2}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \oplus \begin{matrix}\overset{0\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus \\ \underset{0\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \oplus \begin{matrix}\overset{2\quad -2}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus \\ \underset{-2\quad 1}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \oplus \overset{0\quad -1}{\times\!\!=\!\!\Rightarrow\!\!\times} \oplus \overset{-2\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times}.$$

The 1-forms on this homogeneous space G/P are filtered

$$\Lambda^1 = \begin{matrix}\overset{2\quad -2}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus \\ \underset{-2\quad 1}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} + \overset{0\quad -1}{\times\!\!=\!\!\Rightarrow\!\!\times} + \overset{-2\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times} = \begin{matrix}\lambda\\ \oplus\\ \xi\end{matrix} + \lambda\xi + \lambda\xi^2$$

and the corresponding regular infinitesimal flag structure is exactly that of an Engel manifold equipped with a choice of splitting $\Lambda^1_H = \lambda \oplus \xi$ as discussed in §3. The BGG complex (6) in Dynkin diagram notation reads

$$0 \to \mathbb{R} \to \overset{0\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times} \begin{matrix}\nearrow\\ \\ \searrow\end{matrix} \begin{matrix}\overset{2\quad -2}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus\\ \underset{-2\quad 1}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \begin{matrix}\longrightarrow\\ \bowtie\\ \longrightarrow\end{matrix} \begin{matrix}\overset{2\quad -3}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus\\ \underset{-4\quad 1}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \begin{matrix}\longrightarrow\\ \bowtie\\ \longrightarrow\end{matrix} \begin{matrix}\overset{0\quad -3}{\times\!\!=\!\!\Rightarrow\!\!\times}\\ \oplus\\ \underset{-4\quad 0}{\times\!\!=\!\!\Rightarrow\!\!\times}\end{matrix} \begin{matrix}\searrow\\ \\ \nearrow\end{matrix} \overset{-2\quad -2}{\times\!\!=\!\!\Rightarrow\!\!\times} \to 0.$$

§8.   Another geometry in five variables

Recall that an Engel manifold is a 4-dimensional manifold equipped with a generic 2-dimensional distribution. The geometry considered in §3 and §7 was defined on an Engel manifold by a choice of splitting of Λ^1_H, the bundle of 1-forms along H (rather than the filtration that is canonically present). The geometry to be considered in this section will very much resemble this case.

Let us consider a Pfaffian system $I \subset \Lambda^1$ of rank 2 on a smooth 5-manifold. As usual, we define the *Levi form* $\mathcal{L}$ as the composition

$$I \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H,$$

where $H \equiv I^\perp$. Notice that Λ^2_H has rank 3 and we shall suppose that $\mathcal{L}$ is injective, as is generically the case. Under the canonical identification $\Lambda^2_H = \Lambda^3_H \otimes H$ we see that the rank 2 sub-bundle $\mathcal{L}(I) \subset \Lambda^2_H$ gives rise to a rank 2 sub-distribution $D \subset H$. In [13] it is observed that D is the unique rank 2 sub-bundle of H such that $[D, D] \subseteq H$. It is not necessarily the case, however, that $[D, D] = H$ (in which case we would be back in the five variables geometry of §5). To proceed further, let us write L for the line sub-bundle $D^\perp \subset \Lambda^1_H$ and choose a complementary rank 2 sub-bundle Q so that we now have a splitting $\Lambda^1_H = Q \oplus L$. This completes the definition of the structure to be considered in this section.

Equivalently, we are considering a 5-manifold M equipped with a pair of transverse distributions D and ℓ of ranks 2 and 1, respectively, such that

(23) $\qquad [D, D] \subseteq \ell \oplus D \qquad$ and $\qquad [\ell \oplus D, \ell \oplus D] = TM.$

This is precisely the regular infinitesimal flag structure associated with the grading

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{p}} =$$

of $\mathfrak{sl}(4, \mathbb{R})$. The bundles

$$\bigodot{}^c Q \otimes (\Lambda^2 Q)^b \otimes L^a \text{ for } c \in \mathbb{Z}_{\geq 0} \text{ become } \underset{c+2b-2a \quad a-3b-2c \quad c}{\times\!\!-\!\!-\!\!-\!\!\times\!\!-\!\!-\!\!-\!\!\bullet}$$

and the 1-forms are filtered

(24) $\qquad \Lambda^1 = \begin{matrix} Q \\ \oplus \\ L \end{matrix} + I = \underset{-2 \;\; 1 \;\; 0}{\overset{1 \;\; -2 \;\; 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet}} \oplus + \underset{-1 \;\; -1 \;\; 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \, ,$

as expected. This induces filtrations on the higher forms as follows.

and

One can readily verify using an adapted co-frame $\{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5\}$ with

$$I = \mathrm{span}\{\omega^1, \omega^2\}, \quad L + I = \mathrm{span}\{\omega^1, \omega^2, \omega^3\},$$
$$Q + I = \mathrm{span}\{\omega^1, \omega^2, \omega^4, \omega^5\}$$

and such that

$$(25) \quad d\omega^1 \equiv \omega^3 \wedge \omega^4 \bmod \omega^1, \omega^2 \quad \text{and} \quad d\omega^2 \equiv \omega^3 \wedge \omega^5 \bmod \omega^1, \omega^2,$$

that the expected cancellations in the E_0-level of the associated spectral sequence actually take place and we have found a differential complex as follows.

Theorem 7. *On any 5-dimensional manifold equipped with a geo-metric structure defined by transverse distributions D and ℓ of ranks 2 and 1, respectively, and satisfying (23), there is a canonically defined locally exact differential complex*

As in §6, one can make a further normalisation in order to split the two bundles that have arisen from the spectral sequence, or from the equivalent diagram chasing, only as filtered bundles. For the first of these we note that the freedom in its splitting lies in

whereas, from (24), the freedom in splitting Λ^1 lies in

We see that only $\overset{-2\ \ 0\ \ 2}{\times\!\!-\!\!\times\!\!-\!\!\bullet}$ is common to both. Therefore, it is only this freedom that need be eliminated from the freedom to split Λ^1. In fact, once this freedom is eliminated, then Theorem 7 is improved as follows.

Theorem 8. *On any* 5-*dimensional manifold equipped with a regular infinitesimal flag structure defined by* ×—×—• *, there is a canonically defined locally exact differential complex*

$$
0 \to \mathbb{R} \to
\begin{array}{c}
\overset{1\ \ -2\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{-2\ \ 1\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet}
\end{array}
\to
\begin{array}{c}
\overset{2\ \ -3\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{0\ \ -3\ \ 2}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{-3\ \ 0\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet}
\end{array}
\to
\begin{array}{c}
\overset{1\ \ -4\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{-2\ \ -2\ \ 2}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{-4\ \ 0\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet}
\end{array}
\to
\begin{array}{c}
\overset{0\ \ -4\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \\[2pt]
\oplus \\[2pt]
\overset{-3\ \ -2\ \ 1}{\times\!\!-\!\!\times\!\!-\!\!\bullet}
\end{array}
$$
$$
\to \overset{-2\ \ -3\ \ 0}{\times\!\!-\!\!\times\!\!-\!\!\bullet} \to 0.
$$

Proof. As already remarked, to complete the proof we should find a preferred class of splittings of the 1-forms (24) so that the $\overset{-2\ \ 0\ \ 2}{\times\!-\!\times\!-\!\bullet}$ -freedom present in the general splitting is eliminated. As in §6, this can be achieved by restricting a particular component of the exterior derivative $d : \Lambda^2 \to \Lambda^3$ defined via an arbitrary splitting. In this case, we may consider the composition

$$
\overset{0\ \ -2\ \ 0}{\times\!-\!\times\!-\!\bullet} \to \Lambda^2 \xrightarrow{\ d\ } \Lambda^3 \to \overset{-2\ \ -2\ \ 2}{\times\!-\!\times\!-\!\bullet} .
$$

Using an adapted co-framing (25), one may readily verify that
- this is actually a homomorphism of vector bundles,
- insisting that it vanish reduces the freedom in splitting Λ^1 exactly as desired,
- this also eliminates the freedom in splitting the filtered occurring bundles in Theorem 7,

which completes the proof. Q.E.D.

The differential complex in Theorem 8 is our BGG complex for this parabolic geometry. Furthermore, one can easily check that in case $[D, D] = \ell \oplus D$, equivalently if the $\overset{4\ \ -4\ \ 0}{\times\!-\!\times\!-\!\bullet}$ -component of curvature does not vanish, then further cancellations may be effected and one reduces to the BGG complex for the five variables geometry previously discussed in §5.

§9. Pfaffian systems of rank three in seven variables

Let M be a 7-manifold endowed with a generic distribution $H \subset TM$ of rank 4. Equivalently, let $I \subset \Lambda^1$ be a Pfaffian system of rank 3 that is generic in Cartan's sense, meaning that its first derived system I' is zero. We write the corresponding filtration of the cotangent bundle as

$$
\Lambda^1 = \Lambda^1_H + I.
$$

Genericity says that the Levi form, defined as the composition

$$I \to \Lambda^1 \to \Lambda^2 \to \Lambda_H^2,$$

is injective.

It turns out that there exactly two types of generic rank 4 distributions in dimension 7, corresponding to the two open orbits of the action of $\mathrm{GL}(4,\mathbb{R}) \times \mathrm{GL}(3,\mathbb{R})$ on the space of linear maps $\mathrm{Hom}(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ called elliptic, respectively hyperbolic; see [13]. We shall treat these two cases simultaneously.

The Lie algebra $\mathfrak{sp}(6,\mathbb{C})$ admits a grading of the form

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{p}} =$$

$$
\overset{2\quad -1\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
$$
$$\oplus$$
$$
\overset{2\quad 0\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\oplus
\overset{1\quad -1\quad 1}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\oplus
\overset{0\quad -2\quad 2}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\oplus
\overset{1\quad -2\quad 1}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\oplus
\overset{2\quad -2\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}\,.
$$
$$\oplus$$
$$
\overset{0\quad 0\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
$$

There are two real forms of this grading, namely $\mathfrak{sp}(2,1)$ and the split real form $\mathfrak{sp}(6,\mathbb{R})$. One can see that these gradings give rise to an elliptic generic rank 4 distribution on the corresponding 7-dimensional homogeneous space G/P in the first case and to a hyperbolic generic rank 4 distribution on the corresponding homogeneous space G/P in the second case. The parabolic geometries based on these particular G/P are known as *quaternionic contact* [7, §4.3.3] and *split quaternionic contact* [7, §4.3.4], respectively. Regular infinitesimal flag structures of these types correspond exactly to generic rank 4 distributions on 7-manifolds and the irreducible bundles of these geometries can be written as

$$
\overset{a\quad b\quad c}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}\,,
$$

where a, c are non-negative integers. Accordingly, we can write the filtration of the cotangent bundle of a generic rank 4 distribution as

$$
\Lambda^1 = \overset{1\quad -2\quad 1}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet} + \overset{2\quad -2\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
$$

and the filtration of the higher forms as

$$
\Lambda^2 =
\begin{matrix}
\overset{2\quad -2\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet} \\
\oplus \\
\overset{0\quad -3\quad 2}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\end{matrix}
\;+\;
\begin{matrix}
\overset{3\quad -4\quad 1}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet} \\
\oplus \\
\overset{1\quad -3\quad 1}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
\end{matrix}
\;+\; \overset{2\quad -3\quad 0}{\bullet\!-\!\!\times\!\!\Leftarrow\!\bullet}
$$

$$\Lambda^3 = \overset{1\;\;-3\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \begin{array}{c} \overset{4\;\;-4\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{2\;\;-3\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{0\;\;-2\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{2\;\;-5\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} + \begin{array}{c} \overset{3\;\;-5\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{1\;\;-4\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} + \overset{0\;\;-3\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet}$$

$$\Lambda^4 = \overset{0\;\;-2\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \begin{array}{c} \overset{3\;\;-5\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{1\;\;-4\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} + \begin{array}{c} \overset{4\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{2\;\;-4\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{0\;\;-3\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{2\;\;-6\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} + \overset{1\;\;-5\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet}$$

$$\Lambda^5 = \overset{2\;\;-4\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \begin{array}{c} \overset{3\;\;-6\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{1\;\;-5\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} + \begin{array}{c} \overset{2\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{0\;\;-6\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array}$$

$$\Lambda^6 = \overset{2\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \overset{1\;\;-6\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \qquad \Lambda^7 = \overset{0\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet}$$

Choosing an adapted co-framing of the Pfaffian system, one can explicitly verify that all the expected cancellations at the E_0-level of the associated spectral sequence take place and, therefore, one obtains the following.

Theorem 9. *There is a canonically defined locally exact differential complex*

$$0 \to \mathbb{R} \to \overset{1\;\;-2\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \to \begin{array}{c} \overset{0\;\;-3\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ + \\ \overset{3\;\;-4\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} \to \begin{array}{c} \overset{2\;\;-5\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{4\;\;-4\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} \to \begin{array}{c} \overset{4\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ \oplus \\ \overset{2\;\;-6\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} \to$$

$$\begin{array}{c} \overset{3\;\;-6\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \\ + \\ \overset{0\;\;-6\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \end{array} \to \overset{1\;\;-6\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \to \overset{0\;\;-5\;\;0}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} \to 0$$

on any smooth 7-manifold equipped with a generic distribution of rank 4.

The bundles $\overset{0\;\;-3\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \overset{3\;\;-4\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet}$ and $\overset{3\;\;-6\;\;1}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet} + \overset{0\;\;-6\;\;2}{\bullet\!\!\longrightarrow\!\!\times\!\!\Leftarrow\!\!\bullet}$ are sub-quotients of Λ^2, respectively Λ^5. However, there is a preferred class of splittings of the filtration of Λ^1 that canonically splits these bundles as the following result shows.

Theorem 10. *The splittings of the short exact sequence*

$$0 \to \overset{2\;\; -2\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \to \Lambda^1 \to \overset{1\;\; -2\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \to 0$$

are acted freely upon by

$$\mathrm{Hom}\big(\overset{1\;\; -2\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\,,\,\overset{2\;\; -2\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\big) = \overset{3\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \oplus \overset{1\;\; -2\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}.$$

There is a preferred class of splittings in which the $\overset{3\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$ -freedom is eliminated. This restricted choice of splittings canonically splits the bundles $\overset{0\;\; -3\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} + \overset{3\;\; -4\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$ and $\overset{3\;\; -6\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} + \overset{0\;\; -6\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$.

Proof. The only difficulty is in restricting the class of splittings and, as usual, one looks to the exterior derivative $d : \Lambda^2 \to \Lambda^3$ in the presence of a chosen splitting. More specifically, one checks (e.g., in an adapted co-frame) that, having chosen a splitting of Λ^1, the resulting component of $d : \Lambda^2 \to \Lambda^3$ mapping $\overset{1\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$ to $\overset{4\;\; -4\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$ is actually a homomorphism. However,

$$(26)\qquad
\begin{aligned}
\mathrm{Hom}\big(\overset{1\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\,,\,\overset{4\;\; -4\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\big) \;&=\; \overset{1\;\; 0\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \otimes \overset{4\;\; -4\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\\
&\;\|\\
&\;\; \overset{5\;\; -4\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \oplus \overset{3\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}
\end{aligned}$$

and one checks (again using an adapted co-frame or by arguing with irreducible bundles and Schur's lemma) that the original freedom in splitting Λ^1 can be used to eliminate the $\overset{3\;\; -3\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}$ -component. The remaining freedom in splitting Λ^1 is thereby restricted in the appropriate manner (with the stated knock-on effect on the previously identified subquotients of Λ^2 and Λ^5). Q.E.D.

Of course, one could rephrase Theorem 10 as defining the notion of a 2-adapted co-framing and observe that the effect is to reduce the structure bundle of Λ^1 to $P/\exp(\mathfrak{g}_2)$. In any case, combining the two theorems above we immediately obtain the following improved complex.

Theorem 11. *On any smooth 7-manifold endowed with a generic 4-distribution, there is a canonically defined locally exact differential complex*

$$0 \to \mathbb{R} \to \overset{1\;\; -2\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \to \begin{matrix}\overset{0\;\; -3\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\\ \oplus \\ \overset{3\;\; -4\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\end{matrix} \to \begin{matrix}\overset{2\;\; -5\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\\ \oplus \\ \overset{4\;\; -4\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\end{matrix} \to \begin{matrix}\overset{4\;\; -5\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\\ \oplus \\ \overset{2\;\; -6\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\end{matrix} \to$$

$$\begin{matrix}\overset{3\;\; -6\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\\ \oplus \\ \overset{0\;\; -6\;\; 2}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet}\end{matrix} \to \overset{1\;\; -6\;\; 1}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \to \overset{0\;\; -5\;\; 0}{\bullet\!\!\times\!\!\Rightarrow\!\!\bullet} \to 0.$$

Of course, on the homogeneous model G/P this is the standard BGG complex. Finally, as mentioned already in the introduction, let us reiterate that our construction does not see the torsion of this parabolic geometry. In fact, the torsion lies in $\overset{5}{\bullet}\overset{-4}{\longrightarrow}\overset{1}{\times\!\!=\!\!\times}$, which is exactly the component of (26) that we have not eliminated.

§10. The Rumin-Seshadri complex

Although not a replacement for the de Rham complex in resolving the constants, we take the opportunity here to describe another natural differential complex, the Rumin-Seshadri complex [16], the construction of which follows the same general technique. This complex is defined on any symplectic $2n$-manifold M as follows. Denoting by J the symplectic 2-form, let us consider the filtered differential complex $E^\bullet$ defined by

$$E^p = \Lambda^p \oplus \Lambda^{p-1} \text{ for } p = 0, 1, \ldots, 2n+1$$

with differentials

$$(\omega, \mu) \mapsto (d\omega + (-1)^p J \wedge \mu, d\mu).$$

Notice that this complex has local cohomology at both $p = 0$ and $p = 1$. Specifically, the kernel of $E^0 \to E^1$ is $\{(f, 0) \text{ s.t. } f \text{ is constant}\}$ and the cohomology at $p = 1$ is generated by $(\alpha, -1)$, where α is any local potential for the symplectic form J, meaning that $d\alpha = J$. Evidently, this is a filtered complex: the de Rham complex is a sub-complex. The associated spectral sequence immediately gives rise to the Rumin-Seshadri differential complex on M. It has the form

(27)
$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp \xrightarrow{d_\perp} \cdots \xrightarrow{d_\perp} \Lambda^n_\perp$$
$$\downarrow{d^{(2)}_\perp}$$
$$\Lambda^0 \xleftarrow{d_\perp} \Lambda^1 \xleftarrow{d_\perp} \Lambda^2_\perp \xleftarrow{d_\perp} \Lambda^3_\perp \xleftarrow{d_\perp} \cdots \xleftarrow{d_\perp} \Lambda^n_\perp$$

where $\Lambda^p_\perp$ denotes the p-forms that are trace-free with respect to J. The conclusion is as follows.

Theorem 12. *On any symplectic manifold, there is a differential complex* (27) *with local cohomology in degrees 0 and 1. On the level of sheaves, in both these degrees the cohomology is the locally constant sheaf* $\mathbb{R}$. *In all other degrees it is locally exact. On a compact symplectic manifold of dimension ≥ 4,*

(28)
$$\frac{\ker d_\perp : \Gamma(M, \Lambda^1) \to \Gamma(M, \Lambda^2_\perp)}{\operatorname{im} d : \Gamma(M, \Lambda^0) \to \Gamma(M, \Lambda^1)} \cong H^1(M, \mathbb{R}).$$

Proof. The construction of the complex and the identification of its local cohomology are immediate form the spectral sequence. To see (28), note that for a 1-form ω to be in the kernel of $d_\perp$ is to say that $d\omega = fJ$ for some smooth function f but then

$$0 = d^2\omega = df \wedge J \quad \Longrightarrow \quad df = 0 \quad \Longrightarrow \quad f \text{ is constant}$$
$$\Longrightarrow \quad f = 0 \text{ or } J = d(\omega/f).$$

However, the symplectic form cannot be exact for M compact so $f = 0$ and thus $d\omega = 0$. $\hspace{8cm}$ Q.E.D.

In four dimensions, the complex (27) is due to R.T. Smith [17]. In higher dimensions, it was also found by L.-S. Tseng and S.-T. Yau [18] who show that it is elliptic and go on to study its cohomology on compact manifolds. The complex of first order operators after the second-order operator in the middle, was introduced by T. Bouche [3] and who dubbed it the *coeffective* complex (he regarded it as a subcomplex of the second half of the de Rham complex $\Lambda^n \to \cdots \Lambda^{2n}$). The coeffective cohomology was further studied by M. Fernández, R. Ibáñez, and M. de León (see, for example, [11]).

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